Synopsis proposal for Ph.D. programme

Title of the research:

On generalized integral, symmetric integral and trigonometric series

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The problem of recovering a function from it's derivative or to extend the collection of integrable functions is an interesting problem. Newton, Riemann, Lebesgue had defined independently the concept of Newtonian integral, Riemann integral and Lebesgue integral respectively. We know Riemann integrable functions are also Lebesgue integrable. But Newton integral and Lebesgue integral are not comparable. Later Denjoy and Perron defined the concept of Denjoy integral and Perron integral respectively which include Newton integral, Lebesgue integral and improper Riemann integral. A function $f : [a, b] \mapsto \mathbb{R}$ is said to be Denjoy integrable on $[a, b]$ if there exists an ACG_* function $F : [a, b] \mapsto \mathbb{R}$ such that $F^{'} = f$ almost everywhere on $[a, b]$. And the definition of Denjoy integral is a descriptive definition. But definition of Perron integration is constructive. He used to major and minor functions to define Perron integration. A continuous function ψ on $[a, b]$ is said to be major function of $f : [a, b] \mapsto \mathbb{R}$ if $\underline{D}\psi \geq f$ a.e. on $[a, b]$ and $\underline{D}\psi > -\infty$ on $[a.b]$. A continuous function ϕ on $[a, b]$ is said to be minor function of $f : [a, b] \mapsto \mathbb{R}$ if $\overline{D}\phi \leq f$ a.e. on $[a, b]$ and $\overline{D}\phi < \infty$ on $[a.b]$. And he said $f : [a, b] \mapsto \mathbb{R}$ is Perron integrable if $\inf\{\psi(b) - \psi(a) \ : \ \psi \text{ is a major function of } f\} = \sup\{\phi(b) - \phi(a) \ : \ \phi \text{ is a minor function of } f\}$. But Aexandroff and Looman proved that Denjoy integral and Perron integral are equivalent.

Later many authors generalized the scope of the Perron integral by replacing the derivatives occurring in the definition of major and minor functions by generalized derivatives. J. C. Burkill did many considerable work on this topic. Arithmetic mean of $f(x)$ in (a, b), is denoted by $C(f, a, b) = \frac{1}{b-a}\int_a^b f(x)dx$. And a function f is Cesàro derivable if

$$\lim_{h\to 0} \frac{C(f, x, x + h) - f(x)}{\frac{1}{2}h}$$

exists. He defined Cesàro-Perron integral using the Cesàro derivative. J. A. Bergin generalized Perron integral by using Peano derivative. But he also proved tat his integral is equivalent to Burkill's Cesàro-Perron integral.

It is R. E. Svetic who first defined the concept of Laplace derivative. And S. N. Mukhopadhyay and S. Ray established many interesting results for this derivative similar to ordinary derivatives. See [3], [2], [4], [5], [1].

Definition 1 *If f is special Denjoy integrable in a neighbourhood of x. Then f is said to be Laplace derivable at x if for a fixed $\delta > 0$*

$$\lim_{s\to\infty} s^2 \int_o^\delta e^{-st}[f(x + t) - f(x)]dt$$

and

$$\lim_{s\to\infty} (-s^2) \int_o^\delta e^{-st}[f(x - t) - f(x)]dt$$

exists and equal. The common value is called the Laplace derivative of f at x and is denoted by $LD_1 f(x)$.

Definition 2 *If f is special Denjoy integrable in a neighbourhood of x. Then f is said to be Laplace* derivable at x if f is Laplace derivable at x and for a fixed $\delta > 0$*

$$\limsup_{s\to\infty} s^2 \int_o^\delta e^{-st}|f(x+t) - f(x)|dt < \infty \text{ on } [a,b]$$

and

$$\limsup_{s\to\infty} s^2 \int_o^\delta e^{-st}|f(x-t) - f(x)|dt < \infty \text{ on } [a,b].$$

Theorem 1 *Let f is Laplace* derivable at x and g is continuous and Laplace derivable at x. Then $f \cdot g$ is Laplace derivable at x. In addition if we consider g is Laplace* derivable at x then $f \cdot g$ is also Laplace* derivable at x*

A new Perron type of integration called Laplace integration was presented using Laplace derivative. Where we have defined Laplace major function Laplace minor functions in the following way.

Definition 3 (Laplace major function) *Let $f : [a,b] \mapsto \mathbb{R}$ be a function. A Laplace continuous function $\psi : [a,b] \mapsto \mathbb{R}$ is said to be Laplace major function of f if*

- $\underline{LD}_1 \psi \geq f$ *almost everywhere on* $[a,b]$,
- $\underline{LD}_1 \psi > -\infty$ *on* $[a,b]$,
- $\limsup_{s\to\infty} s^2 \int_o^\delta e^{-st}|f(x+t) - f(x)|dt < \infty$ *on* $[a,b]$ *and*
- $\limsup_{s\to\infty} s^2 \int_o^\delta e^{-st}|f(x-t) - f(x)|dt < \infty$ *on* $[a,b]$.

δ is considered positive.

Definition 4 (Laplace minor function) *Let $f : [a,b] \mapsto \mathbb{R}$ be a function. A Laplace continuous function $\phi : [a,b] \mapsto \mathbb{R}$ is said to be Laplace minor function of f if*

- $\overline{LD}_1 \phi \leq f$ *almost everywhere on* $[a,b]$,
- $\overline{LD}_1 \phi < \infty$ *on* $[a,b]$,

- $\limsup_{s\to\infty} s^2 \int_o^\delta e^{-st}|f(x+t)-f(x)|dt < \infty$ *on* $[a,b]$ *and*
- $\limsup_{s\to\infty} s^2 \int_o^\delta e^{-st}|f(x-t)-f(x)|dt < \infty$ *on* $[a,b]$.

δ *is considered positive.*

Definition 5 (Laplace integration) *Let* $f : [a,b] \mapsto \mathbb{R}$ *be a function. Then* f *is said to be Laplace integrable if*

$$\begin{aligned} & \inf\{\psi(b)-\psi(a) \ : \ \psi \text{ is a major function of } f\} \\ = \ & \sup\{\phi(b)-\phi(a) \ : \ \phi \text{ is a minor function of } f\}. \end{aligned}$$

And the common value is denoted by $\int_a^b f$.

After defining the Laplace integral we studied the properties of the integral.

Theorem 2 *Let* $f : [a,b] \mapsto \mathbb{R}$ *be Laplace integrable on* $[a,b]$ *and let*

$$F(x) = \int_a^x f(t)dt \quad x \in [a,b].$$

Then F *is Laplace continuous.*

Theorem 3 *A function* f *which is Laplace integrable is the Laplace derivative of its indefinite Laplace integral almost everywhere.*

Theorem 4 (Fundamental theorem of calculus) *Let* $F : [a,b] \mapsto \mathbb{R}$ *be a function such that* LD_1F *exists finitely on* $[a,b]$ *and* LD_1F *is Laplace integrable on* $[a,b]$. *Then*

$$\int_a^x LD_1F = F(x) - F(a) \qquad \text{for} \qquad x \in [a,b].$$

Theorem 5 (Integration by parts) *Let* f *be Laplace integrable on* $[a,b]$. *And let*

$$F(x) = \int_a^x f(t)dt.$$

If

$$G(x) = \int_a^x g(t)dt$$

where g *is of bounded variation. Then* $f \cdot G$ *is Laplace integrable on* $[a,b]$ *and*

$$\int_a^b f \cdot G = F(b)G(b) - \int_a^b F \cdot g.$$

In future we will try to define descriptive definition of Laplace integral. Which will give relations of BVG_*, ACG_* functions with Laplace derivative and Laplace integrable functions. And if we will get success it will make proofs easier of many important theorems we think.

A symmetric version of Laplace derivative was studied in [6]. See also [7], [8], [9].

Definition 6 *If*

$$\lim_{s\to\infty} s^2 \int_o^\delta e^{-st}[f(x+t) - 2f(x) + f(x-t)]dt = 0$$

we will say f is smooth at x.

And we will denote the upper and lower symmetric second order Laplace derivative as follows.

$$\overline{SLD}_2 f(x) = \limsup_{s\to\infty} s^3 \int_o^\delta e^{-st} \left[\frac{f(x+t) - 2f(x) + f(x-t)}{2}\right] dt$$

and

$$\underline{SLD}_2 f(x) = \liminf_{s\to\infty} s^3 \int_o^\delta e^{-st} \left[\frac{f(x+t) - 2f(x) + f(x-t)}{2}\right] dt.$$

We now define a symmetric Perron type integral using this derivative.

Definition 7 *Let f be an real valued function on $[a,b]$. A function $M : [a,b] \mapsto \mathbb{R}$ will bw called SL^2 major function of f if*

- *ψ is continuous on $[a,b]$,*
- *$LD_1\psi$ exists finitely except on a set of measure zero in (a,b),*
- *$\psi(a) = LD_1\psi(a+) = 0$,*
- *$\underline{SLD}_2\psi \geq f$ a.e. in (a,b),*
- *$\underline{SLD}_2\psi > -\infty$ except on a countable set $E \subset (a,b)$,*
- *M is smooth on E.*

A function $\phi : [a,b] \mapsto \mathbb{R}$ is called a SL^2 minor function of f if $-\phi$ is a SL^2 major function of $-f$.

Definition 8 *With f as above, a function $\psi : [a,b] \mapsto \mathbb{R}$ is called a SL^2_* major function of f if ψ satisfies the first three properties given in the above definition and*

- *$\underline{SLD}_2\psi \geq f$ everywhere in (a,b),*

- $\underline{SLD}_2\psi > -\infty$ *everywhere in* (a, b).

A SL^2_* minor function is defined similarly.

Definition 9 *Let f be an real valued function on $[a, b]$. Then f is said to be SL^2_* integrable on $[a, b]$ if*

$$\sup_{\phi}\{LD_1\phi(b-)\} = \inf_{\psi}\{LD_1\psi(b-)\}.$$

And the common value is denoted by

$$(SL^2_*)\int_a^b f.$$

Also we studied this integral and obtained for results which are given below.

Lemma 1 *If ψ and ϕ are SL^2_* major and SL^2_* minor functions, then $\psi - \phi$ is nonnegative, nondecreasing and convex in $[a, b]$.*

Lemma 2 *For every SL^2 major function ψ of f and every $\epsilon > 0$ there exists a SL^2_* major function ψ_ϵ of f such that*

- $|\psi(x) - \psi_\epsilon(x)| < \epsilon$ *for* $x \in [a, b]$,
- $|LD_1\psi(x) - LD_1\psi_\epsilon(x)| < \epsilon$ *for all* $x \in [a, b]$ *whenever* $LD_1\psi(x)$ *exists.*

Lemma 3 *If ψ and ϕ are SL^2 major and SL^2_* minor functions, then $\psi - \phi$ is nonnegative, nondecreasing and convex in $[a, b]$.*

Theorem 6 *The SL^2 integral and SL^2_* integral are equivalent. The second SL^2 and SL^2_* primitives are also equal.*

Theorem 7 *If f is integrable on $[a, b]$ with its second primitive F, then $SLD_2F = f$ a.e. on (a, b).*

Theorem 8 *If f is integrable on $[a, b]$ then f is measurable and finite a.e., and $M'_{ap} = f$ a.e. in (a, b), where M is the first primitive of f.*

After defining SL^2 integral of a function f we will study its application on the trigonometric series.

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